Hyperbolic Relativity

Joan Massó

NCSA & UIB

BBH GC Alliance meeting - Austin, Oct 95

FOFCH

- The 3+1 Einstein Equations can be written as a **F**irst **O**rder **F**lux **C**onservative **H**yperbolic system

$$\partial_t u + \partial_i F^i(u) = S(u)$$

$$\text{Splitting} \begin{cases} \text{Source} \Rightarrow \partial_t u = S(u) \\ \text{Transport} \Rightarrow \partial_t w + c\,\partial_k w = 0 \end{cases}$$

  - **NO** derivatives in **Fluxes** and nonlinear **Sources**
  - Full set of **Eigenfields** *w*and **characteristic speeds** *c*
    - Bona,Massó, *Phys. Rev. Letters* **68**, 1097 (1992)
    - Bona,Massó,Seidel,Stela, *Phys. Rev. Letters* **75**, 600 (1995)

3D Wave Equation

- The wave equation in second order form

$$\partial_t^2 u \;=\; v^2 \nabla^2 u$$

- written as a FOFCH system: (mimicking 3+1 notation)

$$\partial_t u \;=\; -2K$$

$$\left.\begin{aligned} \partial_t K \;+ v^2 \sum_m \partial_m D_m \;&=\; 0 \\ \partial_t D_k + \partial_k K \;&=\; 0 \end{aligned}\right\} \xrightarrow{\text{dir } k} \left\{\begin{aligned} \partial_t w + c\,\partial_k w &= 0 \\ w \equiv K \pm v D_k \\ c \equiv \pm v \end{aligned}\right.$$

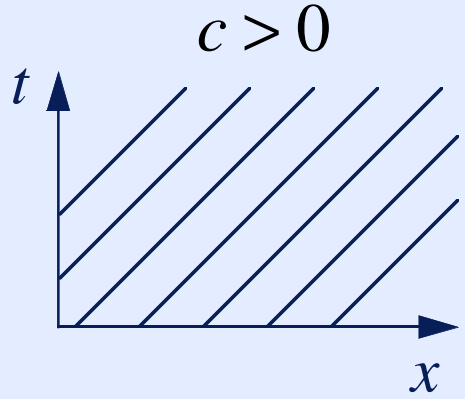
$$\left[D_k \equiv \partial_k u / 2\right] \Leftarrow \text{Initial slice}$$

☛ **Eigenfields** w with **characteristics speeds** c

Numerical Advantages

- Boundary Conditions
  - FOFCH allows to distinguish between left and right moving quantities . For the transport with $c > 0$, only impose left boundary conditions! (and right b.c. for $c < 0$)

$$\partial_t w + c\,\partial_k w = 0$$



- Numerical methods
  - Vast literature for FOFCH (pool of expertise in CFD)
  - Advanced methods for shocks and large gradients

3+1 Einstein Eqs.

- **FOFC** system for **any** lapse and shift: (here for zero shift)

$$\partial_t g_{ij} = -2\alpha K_{ij}$$

$$\partial_t K_{ij} + \partial_k\left(\alpha\, g^{km}\lambda_{mij}\right) = \alpha\, S[K^2, D^2, AD, AV]$$

$$\partial_t D_{kij} + \partial_k\left(\alpha\, K_{ij}\right) = 0$$

$$\partial_t V_i = \alpha\, S[KD, AK]$$

where: $\lambda_{kij} \equiv D_{kij} + g_{k(i}\left(A_{j)} + 2V_{j)} - D_{j)r}{}^{r}\right)$

$$\left[A_k \equiv \partial_k \ln\alpha \quad , \quad D_{kij} \equiv \frac{1}{2}\partial_k g_{ij} \quad , \quad V_k \equiv D_{kr}{}^{r} - D^{r}{}_{rk}\right]$$

☛ the [...] definitions are used only in the initial slice

Einstein Eigensystem

- Eigenfields w and characteristic speeds c: (k direction)

$$\partial_t w + c\partial_k w = 0$$

$$w \equiv g^{kk} K_{ik'} \pm \sqrt{g^{kk}}\,\lambda^k{}_{ik'}$$

$$c_{\text{light}} \equiv \pm\alpha\sqrt{g^{kk}}$$

  - Evolving the eigenfields we cannot recover K (and Ak) unless it is a **given** function.
  - Maximal slicing ($K = 0$) ⇒ **Hyperbolic** system

- For hyperbolicity we need to specify a gauge

"Gauge Dynamics"

- A new class of **hyperbolic** gauge conditions

$$\partial_t \alpha \;=\; -\alpha^2 f\, K$$

  - "missing" eigenfield introduces gauge modes

$$w_{\text{gauge}} \;\equiv\; f g^{kk} K \;\pm\; \sqrt{f g^{kk}}\, \lambda^{km}_{\;\;m}$$

$$c_{\text{gauge}} \equiv \pm \alpha \sqrt{f g^{kk}} = \sqrt{f}\, c_{\text{light}}$$

  - affects Courant condition and Singularity avoidance
  - For Harmonic slicing : $f = 1 \Rightarrow c_{\text{gauge}} = c_{\text{light}}$

- **Maximal** and **Harmonic** are indeed *very* special

Adding a Shift

- Hyperbolic system valid for *any* **given** shift
  - Move all shift derivatives to the sources. Same *w* and

$$c_{\text{zero}} = \beta^k \quad , \quad c_{\text{light}} = \beta^k \pm \alpha\sqrt{g^{kk}}$$

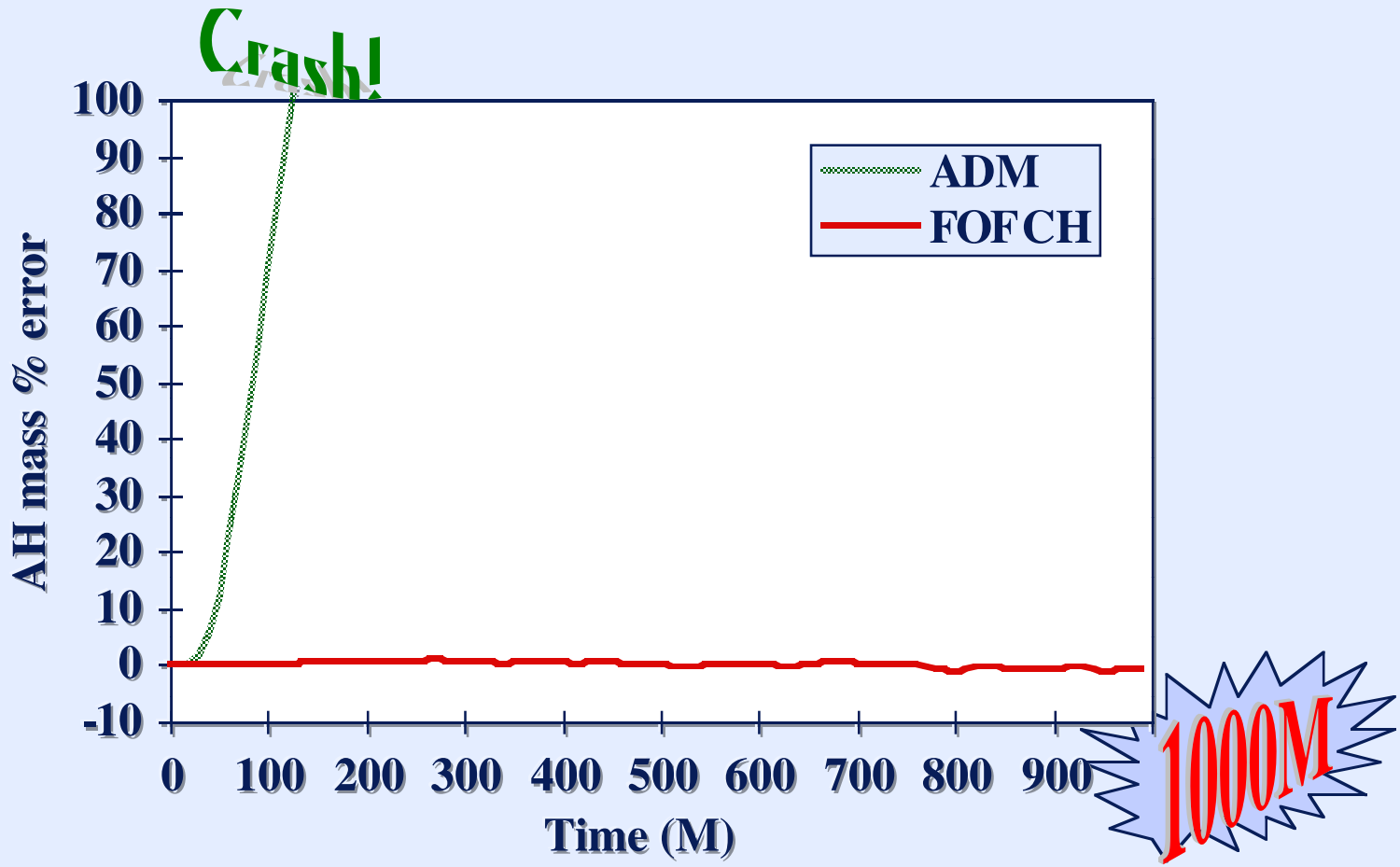
  - Introduce a new time operator (Choquet-Bruhat&York)
- **Dynamic** shift:
  - BMSS : Introduce shift dynamical equation. Difficult!
- **Shift without shift**
  - +Suen: Following S&S *Causal Differencing* for AHBC System+shift ⇨ "tilde" System without shift
  - evolve the transformation functions.

1D Numerical Applications

- 1D Black Hole
  - Harmonic code + Moving grid (i.e., "Evolution of Hyperbolic Einstein Equations using AHBC")
    - BMS, *Phys. Rev. D* **51**, 1639 (1995)
  - New hyperbolic formulation code + TVD methods
    - BMSS, *Phys. Rev. Letters* **75**, 600 (1995)
    - PRD with all the details in preparation
  - + shift using the "Causal Differencing" transformation
    - Work in progress

(• General Relativistic Hydrodynamics
  - Gravitational collapse, etc, shock capturing methods
    - Massó, PhD Thesis (1992), BIMM, Lecture notes 423 (1993))

ADM vs. FOFCH (in an *overly* dramatic comparison)



- Advanced methods (TVD) to treat large gradients

3D New Age code (New H..ha,ha...)

- Full implementation of BMSSS eqs. (and ADM)
  - FOFC for any slicing or shift:
    - MacCormack, Leapfrog, etc for evolving the metric variables
  - FOFCH for the family of algebraic slicings+maximal
    - Upwind, TVD limiter, etc. for evolving the eigenfields.
    - Outgoing boundary conditions.
- Applications
  - **Waves**: Full convergence. Comparisons with G+H using different methods, b.c., eigenevolution, etc..
  - **Black Holes**: Not yet*Isometry*. **AHBC** in progress
- Computational structure: DAGH+vparlib+etc.

Why Hyperbolic Relativity?

Advantages

- Theoretical
  - Hyperbolicity
    - Causal structure
  - Physical Picture
    - "Maxwell vs. GR"
- Numerical Methods
  - CFD arsenal
    - "shock" methods
  - Boundary Conditions
- Physical interpretation
  - Radiation extraction

Disadvantages

- "Complicated"
  - Higher order (CBY 3rd)
  - BMSSS: FO more eqs.
- "Expensive"
  - much more 3D memory
- "Restrictive"
  - Maximal or Harmonic
  - BMSS: other but gauge modes.
- Shift?

Open questions

- Do we need Hyperbolic systems in the BBH Grand Challenge?
  - Does AMR solve the problem of “methods”? any convergent method should suffice?
  - Does Cauchy-Characteristic problem make radiation extraction “unnecessary”? does it “solve” the problem of outgoing boundary conditions?
  - May AHBC be treated in ADM with causal differencing + “suitable” conditions?
  - Do we need “full” control of the causal structure?
- Why not?

Summary

- Hyperbolic Relativity gives deeper mathematical understanting, physical insight and has many numerical analysis advantages.
- Some applications already show this potential but much more needs to be done...
- Some of the major advantages *may* or *may not* be fully relevant for the BBH GC Alliance...
- Stay tuned...