Coordinate Conditions and Their Implementation in 3D Numerical Relativity

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Abstract

We put forth a few ideas on coordinate conditions and their implementation in numerical relativity. Coordinate conditions are important for the long time scale simulations of relativistic systems, e.g., for the determination of gravitational waveforms from astrophysical events to be measured by LIGO/VIRGO. We demonstrate the importance of, and propose methods for, the *active enforcement* of coordinate properties. In particular, the constancy of the determinant of the spatial 3-metric is investigated as such a property. We propose an exceedingly simple but powerful idea for implementing elliptic coordinate conditions that not only makes possible the use of complicated elliptic conditions, but is also *orders of magnitude* more efficient than existing methods for large scale 3D simulations.

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Introduction. It was suggested that the 'Holy Grail of Numerical Relativity' [1] is a 3D code that handles black holes by avoiding singularities, maintains accuracy, and is capable of running forever. We believe, with the apparent horizon boundary condition (AHBC) [2], singularities may no longer be an important threat. Instead, the major difficulty we face, in long time scale evolutions of 3D spacetimes, is to control the coordinate freedom. The aim of this letter is to present a few concepts that are important for formulating, and implementing, good coordinate conditions.

In a numerical simulation the relative motion of coordinate grid points is affected by: (a) gauge choices requiring the metric to take up certain forms (e.g., a diagonal 3-metric); (b) shift vectors which shift the coordinate points from moving normal to the constant time slice; and in particular we want to emphasize, (c) the time slicing condition, which among other things also determines the normal direction of motion of grid points.

The use of gauge conditions of type (a) above is common in lower-dimensional studies, where some symmetries of the spacetime are assumed (e.g., spherical symmetry). In such cases, there are often preferred choices of the form of the 3-metric (e.g., $g_{\theta\theta} = r^2$). However, for a general 3+1 spacetime without any symmetry assumptions, there is *a priori* no preferred choice and we have to make use of the shift and the lapse to control the coordinate degrees of freedom. Both of these are essential for good control.

Time slicing conditions and shift vectors in long time scale 3D evolutions have not been studied much, as previous work in numerical relativity has mostly been carried out for lower-dimensional systems (e.g., axisymmetric systems, where gauge choices in the sense of (a) are effective). Time slicing conditions have been studied, mainly as a tool for avoiding singularities in black hole spacetimes; shift conditions have been used with less success, except in the form of (a) above, and in AHBC work [2]. As we are now forging ahead into full 3D numerical relativity, aiming at long stable evolutions, the choice of the coordinate conditions is crucial.

The long term numerical evolution of full 3D systems is an important goal for numerical relativity, especially for determining the gravitational radiation from realistic astrophysical events that LIGO/VIRGO may detect. For example, for radiation from inspiraling compact binary systems (one of the most important sources expected for LIGO/VIRGO), one would like to be able to numerically evolve the binary system for hundreds or even thousands of M, where M is the mass of the system.

Underlying Ideas. In this paper we study the coordinate conditions in the 3+1 formulation in Cartesian coordinates. The ideas underlying the present study in controlling the lapse α and the shift β^i are the following:

(i) To distinguish good from bad coordinate conditions, one important criterion is: Any *secular changes* in the metric should be due solely to the secular changes of the spacetime geometry, not the gauge degrees of freedom. In this paper, by stable evolutions we mean not just numerical stability but also the absence of "secular drifting" due to gauge freedom.

(ii) In order to obtain a stable evolution over a long time scale, it is important to ensure that the coordinate conditions used are not only suitable for the geometry of the spacetime being evolved, but also that *the conditions themselves are stable.* That is, when the condition is perturbed, e.g., by an inaccuracy in the numerical evolution, there is no long term secular drifting.

(iii) Instead of the lapse, what is more important is the *lapse history.*

(iv) The lapse and shift are interrelated. They should not be considered independently.

(v) An exact implementation of a coordinate condition is unnecessary. An approximate but stable implementation can lead to an *accurate* construction of the spacetime.

Driver Conditions. Here we investigate the *active enforcement* of a coordinate property. The emphasis here is *not* on what properties of the shift or lapse one wants to enforce, but rather the need, and how, to actively enforce a property in 3D numerical evolutions. We begin with a familiar case. The maximal slicing condition [3]

$$K = 0 = \frac{\partial K}{\partial t}, \tag{1}$$

leads to the elliptic equation for the lapse (assuming the Hamiltonian constraint),

$$\nabla^2 \alpha - K_{ij} K^{ij} \alpha - 4\pi (S + \rho) \alpha = 0, \tag{2}$$

where ρ is the energy density and S is the trace of the spatial stress tensor.

To demonstrate the *deficiency* of (2) as a coordinate condition, we first study an evolution of flat space with zero shift and an initial lapse which is close to but not exactly unity

$$\alpha(t=0) = 1.0 + 0.001 \exp[-(x^2+y^2+z^2)]. \tag{3}$$

We evolve the initial data $g_{ij} = \delta_{ij}$ and $K_{ij} = 0$ forward with this lapse for just the first timestep, with subsequent lapses determined by (2). The code used in this study has been described in [4]. The lapse at various times is given in Fig. 1(a). We see that the lapse returns to nearly unity after the first step, as it should. One might expect a healthy evolution. To see what actually happens, in Fig. 1(b) we show the evolution of a typical metric function g_{xx} along the z axis. The dashed lines give g_{xx} up to $t = 8$ at intervals $\Delta t = 2$ under the maximal slicing condition (2). We see that a dip is developing. Eventually this feature is too sharp to be resolved and the code crashes.

The reason behind this development is clear. The maximal slicing Eq. (2) *does not actively enforce* the desired condition $K = 0$. That is, if α is perturbed at any time, making K nonzero, the slicing condition (2) cannot put K back to zero. In Fig. 1(c) the dashed lines show the evolution of K. With the lapse nearly returned to one but the slicing still "curved" (with respect to Minkowski coordinates), this "curvature" will be preserved by the evolution. The grid points moving orthogonal to the slicing drift with respect to one another, leading to secular evolution of the metric functions. This example gives a simple yet vivid demonstration that what is important is not only the lapse but the *history* of the lapse. Although for most of the evolution the lapse is nearly one, the preferred value for flat spacetime, the slicing is bad as errors were made in the past. The value of the lapse at any instant in time does not tell us much about the "shape" of the constant time slice. One needs to control the history.

In this flat space study, the perturbation of the lapse at $t = 0$ is put in by hand. In the actual evolution of 3D relativistic systems involving highly nonlinear and dynamical fields, it is difficult if at all possible to ensure that any coordinate condition is precisely kept everywhere and for all time. Once the condition is perturbed by numerical inaccuracy, without active enforcement, the phenomena shown here can appear.

In the following we show the numerical evolution of the metric functions for an equilibrium compact stellar configuration made up of a self-gravitating scalar field [5]. As the initial configuration has $K = 0$, and is in equilibrium, the metric should remain constant in time with maximal slicing. The actual numerical evolution of g_{rr} (calculated from the Cartesian metric functions g_{xx}, g_{xy}, etc. evolved in the code [4]) for maximal slicing is shown with dashed lines at various times in Fig. 1(d). We see a secular drift, which is caused by K drifting away from zero due to numerical inaccuracy. There is no control over this drifting with maximal slicing.

For all of these cases, we see that the secular drift is related to the fact that Eq. (2) does not actively enforce Eq. (1). Can one construct a slicing condition which *drives* K back to zero when it is perturbed away from it? This leads to what we call the "K-driver" slicing condition

$$\frac{\partial K}{\partial t} + cK = 0, \tag{4}$$

where c is a positive number of our choice. If c is a constant in time, K is driven to zero in an exponential manner. By controlling c, one has control over *the stability of the slicing.* Eq. (4) leads to the elliptic equation for the lapse

$$\nabla^2\alpha - K_{ij}K^{ij}\alpha - \beta^i\nabla_i K - 4\pi(S+\rho)\alpha = cK. \tag{5}$$

The working of the K-driver slicing is shown in Fig. 1 as the solid lines. We see that in each case (Fig. 1(a-d)) the secular drift is under control. There is no assumption of $K = 0$ made, but K is held much closer to zero with (5) than (2) as shown in Fig. 1(c) [6].

We emphasize that the idea of a "driver" is more general than the particular case of (5), which is just one possible way of actively enforcing a coordinate property. Other ways of controlling K are possible, as will be analyzed below, and elsewhere [7] .

The idea of a driver can be applied to other coordinate properties. We next show what we call the γ-driver for the lapse, based on the condition [8]

$$\partial_t(\log\sqrt{\gamma}) = -\alpha K + \nabla_i\beta^i = 0, \tag{6}$$

where $\gamma = \det(\gamma_{ij})$. To actively enforce this "constant γ" condition, we use the "γ-driver slicing"

$$\frac{\partial}{\partial t}\left(-K + \frac{\nabla_i\beta^i}{\alpha}\right) = -c\left(-K + \frac{\nabla_i\beta^i}{\alpha}\right), \tag{7}$$

leading to the elliptic equation for the lapse

$$\begin{aligned} &\nabla^2\alpha - K_{ij}K^{ij}\alpha - \beta^i\nabla_i K - 4\pi(S+\rho)\alpha \\ &\qquad = -\frac{\partial}{\partial t}\left(\frac{\nabla_i\beta^i}{\alpha}\right) + c\left(K - \frac{\nabla_i\beta^i}{\alpha}\right), \end{aligned} \tag{8}$$

where the right-hand side is treated as a source term to be evaluated on the previous slice.

We note that (8) is uncommon as a slicing condition, as it explicitly involves the shift. Traditionally, when considering slicing conditions, the attention is on obtaining a good foliation of the spacetime (hence shift independent). However, we believe that due to the effect of the lapse on the motion of the coordinate grid, foliation is not the only concern. Notice that when the shift is zero, (6) reduces to $K = 0$, the maximal slicing. For a nonzero shift whose divergence has nonzero time-average (e.g., in AHBC [2]), maximal slicing leads to a secular change in γ, cf. (6). The constant γ slicing of (8) does not have this pathology.

The γ-driver slicing (8) is not singularity avoiding, depending on the shift. This is not a drawback, when used with our "AHBC" scheme [2] for handling singularities. In Fig. 2 the solid line gives the evolution of the determinant of the 3-metric γ, for the case of a spherical black hole, obtained using (8) with our "AHBC" code [2]. The shift used is the area-locking shift [2]. This is to be compared to the dashed line, obtained using the same grid parameters but with the K-driver (5). The advantage of the γ-driver in terms of stability against secular drifting of the metric is apparent. Further details of the comparison and the applications of (8) to the evolution of neutron stars and gravitational wave packets will be given in a follow up paper [7].

Elliptic conditions without elliptic equations. In 3D numerical relativity elliptic coordinate conditions (e.g., maximal slicing, or the K and γ-drivers studied above) are often preferred, as they can provide global smoothness. However, elliptic equations are computationally expensive, particularly in 3D with large numbers of grid points. Nonlinear elliptic equations can be complicated and algorithms suitable for solving them are yet to be developed, e.g., for the 3-coupled minimal distortion equations in 3D. Furthermore, complicated boundary conditions, e.g., an irregular shaped inner boundary in the case of determining the shift or lapse with an AHBC, may make a global solution to the elliptic equation very difficult if at all possible.

All these problems can be solved in one stroke, based on the idea that an approximate implementation of the coordinate condition is good enough as long as it is stable. The spacetime constructed can still in principle be exact even when the implementation of the coordinate conditions is approximate.

The idea is to "evolve the elliptic equation" by rewriting it in a parabolic form. For example, instead of directly solving the elliptic equation (5), we take

$$\frac{\partial \alpha}{\partial t} = \epsilon\,(\nabla^2\alpha - K_{ij}K^{ij}\alpha - \beta^i\nabla_i K - 4\pi(S+\rho)\alpha - cK). \tag{9}$$

This is nothing more than the Richardson method [9] of solving elliptic equations, but carried out *in real time as part of the evolution.* As long as dt (the timestep in finite differencing) is much smaller than (i) the dynamical time scales of the spacetime and (ii) the instability timescale (if any) due to the small violation of the coordinate condition, there are many timesteps available for (9) to relax to (5). Note that for implementation of coordinate conditions, the goal is *not* to precisely maintain the condition, but to provide a stable evolution.

Fig. 3(a) shows the evolution of g_{zz} for a 3D Brill wave evolved with the lapse given by (9) and the shift by

$$\begin{aligned}\frac{\partial \beta_i}{\partial t} = \epsilon\,\Big(\nabla^2\beta_i + \frac{1}{3}\nabla_i(\nabla^j\beta_j) + R_i^j\beta_j - \\ \nabla^j\left[2\alpha(K_{ij} - \frac{1}{3}Kg_{ij})\right]\Big)\,,\end{aligned} \tag{10}$$

which is the minimal distortion shift [3], but now implemented through evolution. To the best of our knowledge, this is the first time that the full minimal distortion shift can be implemented in 3D. The result can be compared to the dashed lines obtained using the traditional maximal slicing equation (2) with zero shift. We see that the evolution represented by the solid lines is more stable than that of the dashed lines at late times. The point we want to make here is not really the comparison of stability, but rather that a fully coupled, complicated set of lapse and shift conditions can be implemented with this "evolving elliptic equation" method.

Two technical points are worth mentioning. First, the optimum choice for the value of ϵ is nontrivial [9] and depends on the various time scales involved in the numerical evolution. Second, there are situations where it is desirable to iterate more than once on a single time slice to achieve a more thorough relaxation. As the metric functions are regarded as given on each time slice, the added computational expense is not significant [7].

Recently we have extensively used this "evolving elliptic equations" scheme in implementing coordinate conditions, for our studies of black holes and gravitational waves. The improvement in efficiency, without sacrificing accuracy or stability, can be dramatic, especially when large grids are used. In Fig. 3b the CPU time spent solving maximal slicing (2) using CMSTAB [4](boxes) versus evolving (9) (pluses), is shown for various grid sizes for the Brill wave evolution.

This evolving scheme also makes possible the use of elliptic conditions, e.g., the minimal distortion shift, for black hole studies with a dynamically evolving inner boundary (AHBC). The details of these applications will be reported elsewhere [7]. We thank Joan Massó and Paul Saylor for useful discussions. The calculations in this paper were performed with the CM5 at NCSA and the C90 at PSC. This research is supported by PHY94-04788, PHY94-07882.

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FIGURES

FIG. 1. (a) The lapse at various times is shown. The initial ($t = 0$) lapse is put in by hand. Subsequently we solve either the maximal slicing Eq. (2) (dashed lines) or the K-driver Eq. (5) with $c = 1/(5dt)$ (solid lines) for the lapse. In either case they return to one quickly as expected. (b) g_{xx} at various times is shown. The maximal slicing result shows a secular drift, while the K-driver result is stable. (c) The value of K reaches a non-zero profile with maximal slicing, while the K-driver actively enforces $K = 0$. (d) Here we compare maximal slicing to the K-driver for the numerical evolution of a compact self-gravitating scalar field in equilibrium. In this case, the lapse is perturbed from the exact maximal slicing value only by numerical inaccuracy.

FIG. 2. The evolutions of the determinant of the 3-metric γ for a spherical black hole in the "AHBC" code are compared for the cases of K-driver (dashed lines) vs. γ-driver (solid lines). The runs are done with the same grid parameters. The secular drift is negligible with γ-driver at late time.

FIG. 3. (a) The evolution of the metric component g_{zz} is shown for a Brill wave. The evolution is obtained with maximal slicing Eq. (2) (dashed lines) and by evolving the K-driver Eq. (9) with the minimal distortion shift Eq. (10) (solid lines). $\epsilon = 0.1dx^2/dt$ and $c = 1/(20dt)$ are used. (b) We compare the computational cost of integrating an elliptic lapse condition (boxes) against evolving (pluses) for various sized grids.